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Rotational transformation for reconstruction of digital holography and CGH creation

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Abstract: Rotational transformation allows the reconstruction of images on an arbitrarily tilted plane in a digital holography and the creation of arbitrarily tilted polygon light sources in CGHs. Examples of transformation applications are demonstrated.

OCIS codes: (090.1760) Computer holography; (260.1960) Diffraction theory

1. Introduction

There are various formulations for spatial propagation or the diffraction of coherent light waves in a free space, including such well-known formulas as Fresnel-Kirchhoff integral, Rayleigh-Sommerfeld integral, and Fresnel and Fraunhofer diffraction. However, these share a common restriction: the reference screen in which light fields are obtained as a result of propagation must be parallel to the source plane in which the boundary condition is given. Recently, we reported another formulation for the spatial propagation of light from a source plane to a reference plane [1]. The reference plane is allowed to be arbitrarily tilted to the source plane. We refer to this formula as rotational transformation, which is a useful technique in every field of optical engineering, especially in digital holography and computer-generated holograms (CGH).

A technique of point sources of light is often used to calculate object fields emitted from virtual objects in CGHs for display purposes. This technique is very simple but time-consuming because a gigantic number of point sources must be handled to create an object with a certain dimension. Rotational transformation provides another technique: polygon sources of light instead of point sources [2]. The transformation also solves the hidden-surface removal problem of full-parallax CGHs [3]. Furthermore, rotational transformation can be straightforwardly applied to digital holography. By using this technique, one can reconstruct images on arbitrarily tilted planes from complex-valued images acquired using digital holography as if the plane is parallel to the image sensor.

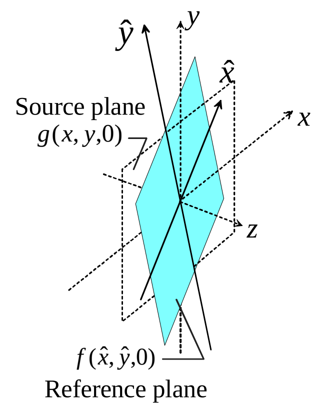

Fig. 1. Definition of source and reference planes

2. Brief proof that rotational transformation satisfies Helmholtz equation

Since discussion has been published on rotational transformation based on the physical interpretation of Fourier frequencies as a wave vector [1], in this report let us describe another proof based on the Helmholtz equation.

Suppose that $\mathbf{r} = (x, y, z)$ and $\mathbf{v} = (u, v, w)$ are the source coordinates system and Fourier frequencies. Reference coordinates system $\hat{\mathbf{r}} = (\hat{x}, \hat{y}, \hat{z})$ shares its origin with source one but is tilted, as shown in Fig. 1. In addition, the Fourier frequencies in reference coordinates are $\hat{\mathbf{v}} = (\hat{u}, \hat{v}, \hat{w})$. Note that Fourier frequencies are not independent of each other, i.e., $w = w(u,v) = (\lambda^{-2} - u^2 - v^2)^{1/2}$, $\hat{w} = \hat{w}(\hat{u}, \hat{v}) = (\lambda^{-2} - \hat{u}^2 - \hat{v}^2)^{1/2}$. Coordinates rotation is given as $\hat{\mathbf{r}} = \mathbf{Tr}$ and $\mathbf{r} = \mathbf{T}^{-1}\hat{\mathbf{r}}$ using rotation matrix $\mathbf{T}$. When distribution of the complex amplitude of a light field are given by $g(x, y, 0)$ in the source plane $(x, y, 0)$, complex amplitude $g(x, y, z)$ in the source coordinates are given by the theory of angular spectrum of plane waves [4] as follows:

$$g(x,y,z) = \mathbf{F}^{-1}\{G(u,v)\exp[i2\pi w(u,v)z]\} = \iint G(u,v)\exp[i2\pi(xu + yv + zw(u,v))]dudv,$$
$$G(u,v) = \mathbf{F}\{g(x,y,0)\} = \iint g(x,y,0)\exp[-i2\pi(ux+vy)]dxdy, \tag{1}$$

where $\mathbf{F}$ and $\mathbf{F}^{-1}$ are Fourier and inverse Fourier transformations, respectively. It is proved that this $g(x, y, z)$ is a strict solution of the Helmholtz equation.

When a rotation matrix is provided, the complex amplitude $f(\hat{x}, \hat{y}, 0)$ on reference plane $(\hat{x}, \hat{y}, 0)$ is written as

$$f(\hat{x},\hat{y},0) = g(a_1\hat{x} + a_2\hat{y}, a_4\hat{x} + a_5\hat{y}, a_7\hat{x} + a_8\hat{y})$$
$$= \iint G(u,v)\exp\left[i2\pi\left\{(a_1u + a_4v + a_7w)\hat{x} + (a_2u + a_5v + a_8w)\hat{y}\right\}\right]dudv, \qquad \mathbf{T}^{-1} \equiv \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \qquad (2)$$

Since $\mathbf{T}={}^t(\mathbf{T}^{-1})$ for any rotation matrix in general, by substituting $a_1u + a_4v + a_7w = \hat{u}$, $a_2u + a_5v + a_8w = \hat{v}$, and $dudv = \left|J(\hat{u},\hat{v})\right|d\hat{u}d\hat{v}$ into Eq. (2), the complex amplitude in the reference plane is rewritten as

$$F(\hat{u},\hat{v}) = G(a_1\hat{u} + a_2\hat{v} + a_3\hat{w}(\hat{u},\hat{v}), a_4\hat{u} + a_5\hat{v} + a_6\hat{w}(\hat{u},\hat{v})), \qquad (3)$$

$$f(\hat{x},\hat{y},0) = \mathbf{F}^{-1}\left\{F(\hat{u},\hat{v})\left|J(\hat{u},\hat{v})\right|\right\},$$
$$J(\hat{u},\hat{v}) \equiv (a_2a_6 - a_3a_5)\hat{u}/\hat{w}(\hat{u},\hat{v}) + (a_3a_4 - a_1a_6)\hat{v}/\hat{w}(\hat{u},\hat{v}) + (a_1a_5 - a_2a_4). \qquad (4)$$

When the parallax condition is fulfilled in the source or reference coordinates, i.e., $u,v << w$ or $\hat{u},\hat{v} << \hat{w}$, Eq. (4) can be reduced to $f(\hat{x},\hat{y},0) \propto \mathbf{F}^{-1}\left\{F(\hat{u},\hat{v})\right\}$. In summary, complex amplitude in the source plane are Fourier-transformed, and then the coordinates rotation of Eq. (3) is performed in Fourier space. Finally, the rotated spectrum is inversely Fourier-transformed to complex amplitude in real space.

3. Reconstruction of tilted planar object in digital holography

Light emitted from a planar object slanted at θ is recorded by lensless-Fourier phase-shifting digital holography [5], as shown in Fig. 2. In this case, $g(x, y, 0)$ is the recorded complex amplitude and $\left|f(\hat{x},\hat{y},0)\right|$ is the reconstructed amplitude image on the tilted plane. Fig. 3 shows the numerical procedure for reconstruction. Spectrum $G(u, v)$ is multiplied by phase factor $p(d) = \exp[i2\pi w(u,v)d]$ for translational propagation prior to coordinates rotation **R** of Eq. (3).

Experimental results are shown in Fig. 4. The number of pixels of the image sensor used for recording is 2000×2000, and the sensor pitch is 6.0×6.0 μm. λ = 532 nm. The distance between the object and image sensor is approximately 19.5 cm. Since lensless-Fourier type digital holography is used, FFT is necessary for obtaining the distribution of complex amplitude. The number of sampling and sampling pitches of the reconstructed images are 2048×2048 and 8.1×8.1 μm, respectively. The planar object is slanted at approximately 70°. Fig. 4(a) is a close-up photograph of the planar object and (b) is reconstruction only using backward translational propagation. The whole object is not emerged due to the depth of focus in both

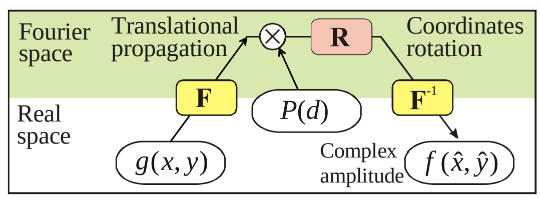

Fig. 3. Numerical procedure for reconstruction of tilted planar object

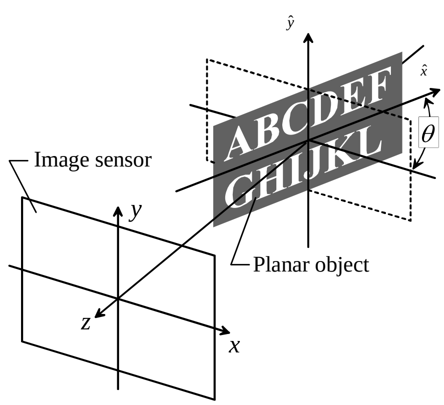

Fig. 2. Geometry for recording and numerical reconstruction of tilted planar object

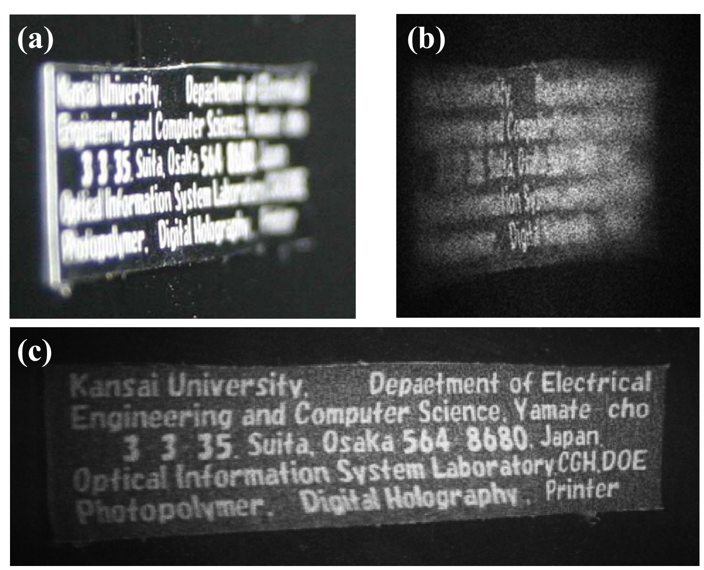

Fig. 4. Photograph of object (a), its numerical reconstruction without (b) and with (c) rotational transformation

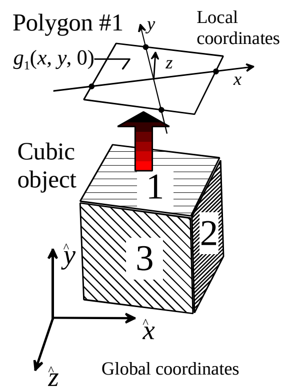

Fig. 5. Example of an object model.

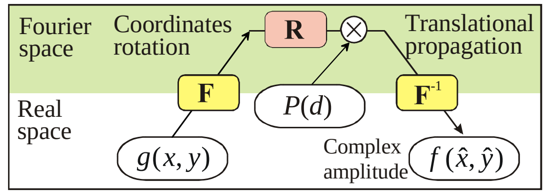

Fig. 6. Numerical procedure for polygon source of light

images, while all characters on the object clearly appear in Fig. 4(c) as a result of rotational transformation at $\theta = 71.8°$.

4. Creation of polygon light source in CGHs

When hidden surfaces are ignored, a cubic object can be resolved into three polygons, as shown in Fig. 5. In CGHs created using rotational transformation, each polygon is regarded as a surface source of light. Total object waves are calculated on the hologram by superimposing object waves emitted from polygons on each other.

Figure 6 shows the procedure for calculating object waves from a polygon. In this case, $g(x, y, 0)$ is given for every polygon in the local coordinates, which are also defined for each polygon. An example of local coordinates is shown in Fig. 5. $g(x, y, 0)$ is designed so that its amplitude distribution forms a polygon and its phase distribution plays the role of a diffuser [2]. This $g(x, y, 0)$ is referred to as the property function of the polygon.

The property function of polygon #1 is shown in Fig. 7(a) as an example. The local coordinates of the property function are rotated to be parallel to the hologram after performing FFT (Fig. 7(c)), and then the complex amplitude is calculated by inverse FFT after translational propagation in Fourier space.

An example of the optical reconstruction of a CGH created by these procedures is shown in Fig. 8. The number of pixels is 8192×4096, and pixel pitches are 2×4 μm. $\lambda = 633$ nm.

5. Conclusion

Rotational transformation strictly satisfies Helmholtz's equation and makes it possible to calculate the distribution of complex amplitude on arbitral planes not parallel to the source plane. This technique can be applied to both digital holography and CGHs.

6. References

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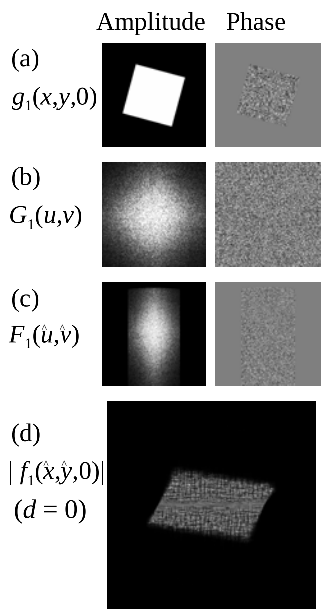

Fig. 7. Process of creating polygon #1

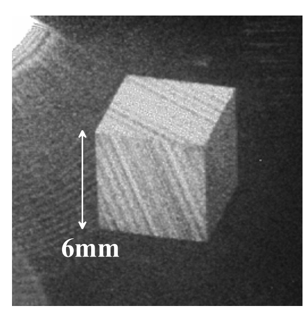

Fig. 8. Optical reconstruction of CGH created by polygon light sources